

Low Cycle Fatigue: Physics and Equations for Review

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lcf-strain-life. Both authors contributed equally.

July 23, 2026

Abstract

This is a self-contained statement of the fatigue physics implemented in the toolkit, written for review by a materials or fatigue specialist. It contains the equations, the symbols and units, the conventions, and the source for each relation. It contains no software detail. A short sign-off table is at the end.

1 Conventions

All analysis uses true stress and true strain. Stress and modulus are in MPa. Strain is a dimensionless fraction. Life is expressed in reversals $2N_f$, where one cycle is two reversals. The fatigue strength exponent b and the fatigue ductility exponent c are negative.

Symbol	Meaning	Unit
σ, ε	true stress, true strain	MPa, -
$\Delta\sigma/2, \Delta\varepsilon/2$	stress, total strain amplitude	MPa, -
$\Delta\varepsilon_e/2, \Delta\varepsilon_p/2$	elastic, plastic strain amplitude	-
σ_m, σ_{max}	mean stress, maximum stress	MPa
R, R_ε	stress ratio, strain ratio	-
E	Young's modulus	MPa
σ'_f, b	fatigue strength coefficient, exponent	MPa, -
ε'_f, c	fatigue ductility coefficient, exponent	-
K', n'	cyclic strength coefficient, strain-hardening exponent	MPa, -
$2N_f$	reversals to failure	-
K_t, K_f, q	stress concentration, fatigue notch factor, notch sensitivity	-

2 True stress and strain

Engineering values e and $\sigma_{\text{eng}} = F/A_0$ convert to true values by

$$\varepsilon = \ln(1 + e), \quad \sigma = \sigma_{\text{eng}}(1 + e). \quad (1)$$

Valid up to necking. Reference: Dowling, Mechanical Behavior of Materials, 4th ed.

3 Cyclic stress-strain, Ramberg-Osgood

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{1/n'}, \quad \Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2K'}\right)^{1/n'}. \quad (2)$$

The second form is the doubled hysteresis branch. Reference: Ramberg and Osgood 1943, NACA TN 902. Dowling 4th ed., Eq. 14.12.

4 Strain-life

Elastic, Basquin:

$$\frac{\Delta\sigma}{2} = \sigma'_f(2N_f)^b, \quad \frac{\Delta\varepsilon_e}{2} = \frac{\sigma'_f}{E}(2N_f)^b. \quad (3)$$

Plastic, Coffin-Manson:

$$\frac{\Delta\varepsilon_p}{2} = \varepsilon'_f(2N_f)^c. \quad (4)$$

Total strain-life and the elastic-plastic transition life:

$$\frac{\Delta\varepsilon}{2} = \frac{\sigma'_f}{E}(2N_f)^b + \varepsilon'_f(2N_f)^c, \quad 2N_t = \left(\frac{\varepsilon'_f E}{\sigma'_f}\right)^{1/(b-c)}. \quad (5)$$

Compatibility, checked and reported but not forced:

$$n' = \frac{b}{c}, \quad K' = \frac{\sigma'_f}{(\varepsilon'_f)^{b/c}}. \quad (6)$$

The plastic line is fit over the low cycle regime, excluding near-runout points whose plastic strain is at measurement noise level. References: Basquin 1910, Coffin 1954, Manson 1953, Dowling 4th ed. Eq. 14.3 to 14.6.

5 Mean-stress corrections

The asymmetry of a constant amplitude cycle is the stress ratio $R = \sigma_{min}/\sigma_{max}$, or the strain ratio $R_\varepsilon = \varepsilon_{min}/\varepsilon_{max}$ under strain control. Strain-controlled testing is the norm in the low cycle regime per ASTM E606 and ISO 12106, stress control is the norm in the high cycle regime.

Ratio	Loading	Minimum load	Maximum load
$R = -1$	fully reversed, the LCF baseline	compression	equal tension
$R = 0$	pulsating tension	zero	tension
$0 < R < 1$	tension-tension	tension	higher tension
$R = \pm\infty$	pulsating compression	compression	zero
$1 < R < \infty$	compression-compression	higher compression	compression

The control mode decides what a nonzero mean does over the life of a test. Under strain control the mean stress relaxes toward zero as plastic strain accumulates, and Morrow suits balanced or compressive means while SWT suits tensile means. Under stress control the specimen ratchets, accumulating strain in the direction of the mean stress, and the Walker fitted exponent captures

the stress ratio sensitivity. The corrections in this section apply to the stabilized cycle. Reference: Dowling 4th ed. ch. 9.

The cycle-dependent evolution is modeled separately. Mean stress relaxation under strain control follows $\sigma_m(N) = \sigma_{m,1}N^{b_r}$ with $b_r \leq 0$. Ratcheting under stress control accumulates strain as $\varepsilon_r(N) = CN^p$, and its life interaction is a ductility-exhaustion penalty on the plastic strain-life line, $\Delta\varepsilon_p/2 = (\varepsilon'_f - \varepsilon_r)(2N_f)^c$. These forms were reconstructed from collaborator notes and match the standard published forms, pending the collaborator's confirmation. References: Morrow and Sinclair 1958 (ASTM STP 237), Jhansale and Topper 1973 (ASTM STP 519), Xia, Kujawski and Ellyin 1996 (Int. J. Fatigue 18:335), Kapoor 1994 (Fatigue Fract. Eng. Mater. Struct. 17:201).

Morrow, elastic term shifted by the mean stress:

$$\frac{\Delta\varepsilon}{2} = \frac{\sigma'_f - \sigma_m}{E}(2N_f)^b + \varepsilon'_f(2N_f)^c. \quad (7)$$

Modified Morrow, both terms shifted:

$$\frac{\Delta\varepsilon}{2} = \frac{\sigma'_f - \sigma_m}{E}(2N_f)^b + \varepsilon'_f \left(\frac{\sigma'_f - \sigma_m}{\sigma'_f} \right)^{c/b} (2N_f)^c. \quad (8)$$

Smith-Watson-Topper:

$$\sigma_{max} \frac{\Delta\varepsilon}{2} = \frac{(\sigma'_f)^2}{E}(2N_f)^{2b} + \sigma'_f \varepsilon'_f (2N_f)^{b+c}, \quad \sigma_{ar} = \sqrt{\sigma_{max} \sigma_a}. \quad (9)$$

Walker, with the equivalent fully-reversed amplitude and the steel estimate of the exponent:

$$\sigma_{ar} = \sigma_{max}^{1-\gamma} \sigma_a^\gamma, \quad \gamma_{steel} = 0.8818 - 2.00 \times 10^{-4} \sigma_u. \quad (10)$$

Morrow equivalent fully-reversed amplitude:

$$\sigma_{ar} = \frac{\sigma_a}{1 - \sigma_m/\sigma'_f}. \quad (11)$$

With $\gamma = 0.5$ Walker reduces to SWT. References: Morrow 1968. Smith, Watson, Topper 1970, J. Materials 5(4):767. Walker 1970. Dowling, Calhoun, Arcari 2009, Fatigue Fract. Eng. Mater. Struct. (steel γ). Dowling 4th ed. Eq. 9.18 to 9.21.

6 Hysteresis energy and cyclic response

The plastic strain energy density per cycle is the closed loop area

$$W = \oint \sigma d\varepsilon. \quad (12)$$

The tension-compression asymmetry of a cycle is $R_{TC} = |\sigma_{max}|/|\sigma_{min}|$. Peak and valley stress versus cycle give the cyclic hardening and softening response.

7 Variable amplitude, cycle counting

Irregular histories are reduced to closed cycles by the rainflow method, which pairs reversals into hysteresis loops while preserving their order in time. Reference: ASTM E1049-85(2017), Downing and Socie 1982, Matsuishi and Endo 1968.

Level-crossing counting records positive-slope crossings at and above a reference level and negative-slope crossings below it. Peak counting records peaks at and above the reference and valleys below it. Both follow ASTM E1049-85(2017), sections 5.2 and 5.3.

The racetrack (gate) filter condenses a history before counting by removing swings smaller than a gate while keeping the order of the large reversals. Reference: Fuchs, Nelson, Burke, and Toomay, SAE paper 730565, 1973.

8 Variable-amplitude local strain simulation

A repeating strain history block is rotated to its global maximum and its stress response simulated: the initial loading follows the cyclic Ramberg-Osgood curve, every subsequent branch follows the doubled hysteresis form given above from its reversal origin, and material memory follows the rainflow closure rule, so a closed interior loop leaves the outer branch exactly where it would have been without the interruption. Each closed loop carries its strain amplitude and its simulated peak and mean stress, its life comes from the SWT, Morrow, or uncorrected strain-life relation above, and the damage is Miner-summed to blocks to failure. Sequence effects enter through the loop means: a small cycle riding on a large branch carries the mean stress of its position.

A load-input mode covers notched members: given a nominal stress history and K_t , the initial loading follows Neuber's rule on the cyclic curve and every branch follows the modified Neuber rule on the doubled curve, so the loops carry the local notch-root strain and stress.

Assumptions, stated plainly: stabilized cyclic properties throughout, no cycle-dependent mean stress relaxation, no ratcheting. The simulation is consistent with the constant-amplitude solvers and with rainflow counting by construction and by test. Validation, strain input: against the published SAE smooth-specimen dataset of Conle (MSc thesis, University of Waterloo, 1974, data distributed by the SAE FD and E committee), predictions fall within a factor of two of experiment for the transmission and bracket histories and about a factor of three, non-conservative, for the suspension history, consistent with the documented scatter of linear-damage local-strain predictions on this program. Validation, load input: against the SAE keyhole benchmark (AE-6, Wetzel ed., 1977, inputs and results preserved on the Internet Archive copy of the eFatigue benchmark page), the constant-amplitude RQC-100 case predicts within 4 percent of the benchmark's own strain-life calculation and conservatively against the crack-based experimental life, and the Man-Ten suspension variable-amplitude case predicts within a factor of two of the three experimental lives. References: Masing 1926 (Proc. 2nd Int. Congress for Applied Mechanics, Zurich), Dowling 4th ed. ch. 14, ASTM E1049-85(2017), Neuber 1961, Conle 1974 and the SAE FD and E archive via fde.uwaterloo.ca.

9 Cumulative damage

Palmgren-Miner linear damage, failure at a critical sum, the default being one:

$$D = \sum_i \frac{n_i}{N_{f,i}}. \quad (13)$$

Double Linear Damage Rule, Manson-Halford, with the Phase I life fraction referenced to the longest life in the spectrum:

$$f_I = 0.35 \left(\frac{N_f}{N_{long}} \right)^{0.25}, \quad N_I = N_f f_I, \quad N_{II} = N_f (1 - f_I). \quad (14)$$

Phase I accumulates to one, then Phase II accumulates to one. Corten-Dolan, where σ_1 is the maximum stress in the block and α_i the cycle fractions:

$$N = \frac{N_{f,1}}{\sum_i \alpha_i (\sigma_i / \sigma_1)^d}. \quad (15)$$

References: Palmgren 1924, Miner 1945, Manson and Halford 1981 (Int. J. Fracture 17:169), Corten and Dolan 1956.

For stress-based collectives the allowable life comes from a one-slope Woehler line with a knee at (S_D, N_D) :

$$N = N_D \left(\frac{S_a}{S_D} \right)^{-k} \quad (S_a \geq S_D), \quad (16)$$

and below the knee one of three treatments: infinite life (Miner original), the same slope k continued (Miner elementary), or the flatter fictitious slope $2k - 1$ (Haibach):

$$N = N_D \left(\frac{S_a}{S_D} \right)^{-(2k-1)} \quad (S_a < S_D). \quad (17)$$

References: Miner 1945, Haibach 1970, described in Haibach, Betriebsfestigkeit, 3rd ed., Springer, 2006.

10 Notch local-strain approach

Neuber, combined with the cyclic curve, and its range form:

$$\frac{(K_t S)^2}{E} = \sigma \varepsilon, \quad \frac{(K_t \Delta S)^2}{E} = \Delta \sigma \Delta \varepsilon. \quad (18)$$

Glinka equivalent strain energy density:

$$\frac{(K_t S)^2}{2E} = \frac{\sigma^2}{2E} + \frac{\sigma}{n' + 1} \left(\frac{\sigma}{K'} \right)^{1/n'}. \quad (19)$$

Fatigue notch factor and notch sensitivity:

$$K_f^{\text{Peterson}} = 1 + \frac{K_t - 1}{1 + a/r}, \quad K_f^{\text{Neuber}} = 1 + \frac{K_t - 1}{1 + \sqrt{\beta/r}}, \quad q = \frac{K_f - 1}{K_t - 1}. \quad (20)$$

Neuber tends to overestimate and Glinka to underestimate the local strain. The measured value usually lies between them. References: Neuber 1961, Molski and Glinka 1981, Peterson 1974.

11 Statistics and design curves

Life is the dependent variable in the linearized regression, $\log_{10} N = A + B \log_{10}(\Delta\varepsilon/2)$. Confidence and prediction intervals use the residual standard error and the Student t quantile. A reliability and confidence design curve is the mean reduced by $k s$, where k is the one-sided Owen tolerance factor. Right-censored runouts are handled by maximum likelihood rather than deletion. References: ASTM E739, withdrawn 2024 and used as the de facto reference, Owen 1963, Williams, Lee and Rilly 2003 (Int. J. Fatigue 25:427).

The censored maximum-likelihood layer follows the framework behind the ASTM replacement work item WK88010. With $y_i = \log_{10} N_i$, $x_i = \log_{10}(\Delta\varepsilon/2)_i$, location $\mu_i = \beta_0 + \beta_1 x_i$, scale σ , and standardized residual $z_i = (y_i - \mu_i)/\sigma$, observed failures contribute the density and runouts the survival probability,

$$\ell(\beta_0, \beta_1, \sigma) = \sum_{\text{failures}} \left[\ln \phi(z_i) - \ln \sigma \right] + \sum_{\text{runouts}} \ln S(z_i), \quad (21)$$

where ϕ and S are the density and survival function of the standardized scatter family: the standard normal for lognormal life, or the smallest-extreme-value distribution, $S(z) = \exp(-e^z)$, for Weibull life. Standard errors come from the observed information, the inverse Hessian of $-\ell$ at the optimum, and the families are compared by AIC. The reliability- p life quantile at an amplitude x_0 is $q_p = \beta_0 + \beta_1 x_0 + z_p \sigma$ with z_p the standardized quantile of the chosen family. Its one-sided lower confidence bound at level γ inverts the likelihood ratio by profile likelihood,

$$q_L = \min \left\{ q : 2[\ell(\hat{\theta}) - \ell_p(q)] \leq z_\gamma^2 \right\}, \quad (22)$$

with ℓ_p the profile log likelihood over the remaining parameters, or alternatively the Wald bound $\hat{q} - z_\gamma \text{se}(\hat{q})$ with the delta method. The Owen construction assumes a complete normal sample, so with censoring the profile bound is the one to prefer, and on complete samples the two agree closely, which the test suite checks. The same likelihood fits the combined strain-life curve directly, with $\mu(\varepsilon_a)$ the base-10 logarithm of the life obtained by inverting the total strain-life equation, which the withdrawn E739 could not represent, its scope was linearized fits. The four constants of the combined curve are strongly correlated when inferred from total strain alone and the elastic exponent can collapse toward zero on sparse data. The fitted curve stays well determined inside the tested strain range, the standard errors report the weakness, and results carry an identifiability warning. References: Meeker, Escobar, Pascual, Hong, Liu, Falk and Ananthasayanam, arXiv:2212.04550, Meeker and Escobar 1998 (Statistical Methods for Reliability Data, Wiley), Venzon and Moolgavkar 1988 (Applied Statistics 37:87). Validation: exact agreement with least squares in the uncensored lognormal limit including the $\sqrt{(n-2)/n}$ scale relation, seeded synthetic recovery at curve level, and profile-versus-Owen consistency on complete samples.

The fatigue limit from a staircase (up-and-down) test is estimated by the Dixon-Mood method. With n_i the counts of the less frequent event on the level grid, $A = \sum i n_i$, $B = \sum i^2 n_i$, $N = \sum n_i$, step d , and X_0 the lowest level where that event occurred,

$$\hat{\mu} = X_0 + d \left(\frac{A}{N} \mp \frac{1}{2} \right), \quad \hat{\sigma} = 1.62 d \left(\frac{NB - A^2}{N^2} + 0.029 \right) \quad \text{for} \quad \frac{NB - A^2}{N^2} \geq 0.3, \quad (23)$$

with the minus sign when the analysis uses failures and the plus sign for survivals. Below the 0.3 variability bound the standard deviation is the approximate fallback $\hat{\sigma} = 0.53 d$ and is flagged. A-basis and B-basis values are the one-sided lower tolerance bounds $\bar{x} - k s$ with the Owen factor

at 99 percent reliability, 95 percent confidence and 90/95 respectively. When the data contain replicate amplitude levels the linear fit is checked by the lack-of-fit F test,

$$F = \frac{SS_{LOF}/(m-2)}{SS_{PE}/(n-m)}, \quad (24)$$

where m is the number of distinct levels. References: Dixon and Mood 1948 (J. Amer. Statist. Assoc. 43:109), ISO 12107:2012, Owen 1963 (Sandia SCR-607), ASTM E739. Validated against the S34MnV staircase example of Ekaputra et al. 2020 (Open Engineering 10:394).

The random fatigue limit model treats each specimen's fatigue limit γ as unit-to-unit random, $V = \log \gamma \sim N(\mu_\gamma, \sigma_\gamma)$, with log life conditionally normal,

$$W = \log N \mid V \sim N(\beta_0 + \beta_1 \log(s - \gamma), \sigma), \quad s > \gamma, \quad (25)$$

so the S-N curve flattens naturally near the limit and runouts enter the likelihood as censored observations, including the probability that the limit sits above the test stress. The five parameters are fit by maximum likelihood with the marginal integral over V evaluated by quadrature. Validation: the fitter reproduces the published Pascual-Meeker normal-normal fit of the laminate-panel dataset exactly, log-likelihood -86.221 and parameters matching their Table 1 to the digit, with the likelihood also cross-checked against brute-force integration. Reference: Pascual and Meeker 1999 (Technometrics 41:277), Meeker et al. 2026 (Statistical Science 41:1).

Outlier screening operates on the residuals of the log-life regression. Runouts are censored observations, not outliers, and are excluded from the screen. A single suspect point uses the two-sided Grubbs test, several suspect points use the generalized extreme studentized deviate test, whose approximation is intended for $n \geq 15$. Influence is reported through leverage, externally studentized residuals against a Bonferroni-corrected t critical value, and Cook's distance against the $4/n$ screening threshold. Flagged points are reported, never deleted automatically. References: Grubbs 1969 (Technometrics 11:1), Rosner 1983 (Technometrics 25:165), Cook 1977 (Technometrics 19:15), NIST/SEMATECH e-Handbook of Statistical Methods, sections 1.3.5.17.1 and 1.3.5.17.3.

12 Estimation of strain-life constants

When no strain-controlled test data exists, the four constants are estimated from monotonic properties. S_u is the ultimate tensile strength in MPa, HB the Brinell hardness, RA the reduction in area as a fraction, and $\tilde{\epsilon}_f = \ln[1/(1 - RA)]$ the true fracture ductility. Estimates are screening values, not substitutes for test data.

Medians method, steels (recommended default) and aluminum alloys:

$$\begin{aligned} \text{steel: } & \sigma'_f = 1.5 S_u, \quad b = -0.09, \quad \epsilon'_f = 0.45, \quad c = -0.59, \\ \text{aluminum: } & \sigma'_f = 1.9 S_u, \quad b = -0.11, \quad \epsilon'_f = 0.28, \quad c = -0.66. \end{aligned} \quad (26)$$

Uniform Material Law, steels, with the ductility correction $\psi = 1$ for $S_u/E \leq 0.003$, else $\psi = 1.375 - 125 S_u/E$:

$$\sigma'_f = 1.50 S_u, \quad b = -0.087, \quad \epsilon'_f = 0.59 \psi, \quad c = -0.58, \quad K' = 1.65 S_u, \quad n' = 0.15, \quad (27)$$

and for aluminum and titanium alloys $\sigma'_f = 1.67 S_u$, $b = -0.095$, $\epsilon'_f = 0.35$, $c = -0.69$. The law loses validity as S_u approaches 2.2 GPa, where ψ reaches zero.

Universal slopes, any metal:

$$\sigma'_f = 1.9 S_u, \quad b = -0.12, \quad \epsilon'_f = 0.76 \tilde{\epsilon}_f^{0.6}, \quad c = -0.6. \quad (28)$$

Modified universal slopes, steels:

$$\sigma'_f = 0.623 E \left(\frac{S_u}{E} \right)^{0.832}, \quad b = -0.09, \quad \varepsilon'_f = 0.0196 \left(\frac{S_u}{E} \right)^{-0.53} \tilde{\varepsilon}_f^{0.155}, \quad c = -0.56. \quad (29)$$

Hardness method, steels with roughly 150 to 700 HB:

$$\sigma'_f = 4.25 HB + 225, \quad b = -0.09, \quad \varepsilon'_f = \frac{0.32 HB^2 - 487 HB + 191000}{E}, \quad c = -0.56. \quad (30)$$

References: Meggiolaro and Castro 2004 (Int. J. Fatigue 26:463, also the comparative evaluation over 845 metals), Baeumel and Seeger 1990, Manson 1965 (Exp. Mech. 5:193), Muralidharan and Manson 1988 (J. Eng. Mater. Technol. 110:55), Roessle and Fatemi 2000 (Int. J. Fatigue 22:495).

13 Elevated temperature

Frequency-modified Coffin-Manson, coefficient form:

$$C_f = C_o \left(\frac{f}{f_{ref}} \right)^{k-1}, \quad \frac{\Delta \varepsilon_p}{2} = C_f (2N_f)^c. \quad (31)$$

Linear time-fraction creep-fatigue damage, fatigue plus creep:

$$D = \sum_i \frac{n_i}{N_{f,i}} + \sum_j \frac{t_j}{t_{r,j}}, \quad (32)$$

checked against a bilinear creep-fatigue interaction envelope. References: Coffin 1971, Robinson 1952, ASTM E2714-13(2020).

14 Multiaxial critical plane

Critical-plane parameters:

$$P_{FS} = \frac{\Delta \gamma_{max}}{2} \left(1 + k \frac{\sigma_{n,max}}{\sigma_y} \right), \quad P_{BM} = \frac{\Delta \gamma_{max}}{2} + S \Delta \varepsilon_n, \quad P_{SWT} = \sigma_{n,max} \frac{\Delta \varepsilon_1}{2}. \quad (33)$$

The plane search takes strain and stress tensor component histories sampled over one cycle and scans plane normals over a hemisphere grid. Per plane, the normal strain history is $n \cdot \varepsilon \cdot n$ with amplitude half its range, the shear amplitude is half the longest chord of the in-plane shear-vector path, which stays meaningful for non-proportional paths, and the maximum normal stress is the maximum of $n \cdot \sigma \cdot n$ over the cycle. Scope, stated plainly: amplitudes come from the given cycle's path, per-plane rainflow counting of long multiaxial histories is not implemented. Validated against the closed forms for uniaxial loading with Poisson contraction (45 degree plane, engineering shear amplitude $(1 + \nu) \varepsilon_a$) and pure torsion. References: Fatemi and Socie 1988 (Fatigue Fract. Eng. Mater. Struct. 11(3):149), Brown and Miller 1973, Smith, Watson, Topper 1970, Socie and Marquis, Multiaxial Fatigue, SAE, 2000.

15 Conventions and assumptions to confirm

- True stress-strain conversion assumed valid up to necking.
- Plastic strain amplitude taken as $\Delta\varepsilon_t/2 - \Delta\sigma/(2E)$.
- Failure criterion is a percent load drop from the stabilized half-life peak load, default 30 percent.
- Default mean-stress model for variable amplitude is SWT, notch default is Neuber, damage default is Miner.
- The plastic strain-life line is fit over the low cycle regime only.

16 Reviewer sign-off

Reviewer name and date, then a verdict per section. Suggested verdicts: OK, OK with note, or needs change.

Section	Verdict	Note
True stress and strain		
Ramberg-Osgood		
Strain-life		
Mean-stress corrections		
Energy and cyclic response		
Cycle counting and filtering		
Variable-amplitude simulation		
Cumulative damage and knee variants		
Notch local-strain		
Statistics, design curves, outlier screen		
Estimation of strain-life constants		
Elevated temperature		
Multiaxial critical plane		
Conventions and assumptions		